9.3.3 The Method of Eigenfunction Expansions for Green's function.

Consider the nonhomogeneous BVP

$$\begin{cases} -u''(x) = f(x), & 0 < x < L \quad (1.1) \\ u(0) = 0, \quad u(L) = 0 & \quad (1.2) \end{cases}$$

We want to derive the Greens function, $G(x, x_0)$ associated to this differential operator

$$L \equiv \frac{-d^2}{dx^2} \qquad (1.3)$$

This is done representing G in terms of the eigenfunctions of the associated eigenvalue problem (EVP):

$$\begin{cases} -\phi''(x) = \lambda \phi(x), & 0 < x < L \quad (1.4) \\ \phi(0) = 0, \quad \phi(L) = 0. & \quad (1.5) \end{cases}$$

Equ. (1.4) can be written as

$$\phi''(x) + \lambda \phi(x) = 0$$

and we already know from MSV. that

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, \ldots \tag{2.1}$$

To obtain the Green's fn. in terms of these eigenfunctions, we expand $u(x)$ and $f(x)$

$$u(x) = \sum_{n=1}^{\infty} u_n \phi_n(x), \quad f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \tag{2.2}$$

Substituting (2.2) into (1.1)

$$-\left(\sum_{n=1}^{\infty} u_n \phi_n(x)\right)'' = \sum_{n=1}^{\infty} f_n \phi_n(x) \tag{2.3}$$

Now, $\phi_n(x)$ and $u_n(x)$ satisfy the same homog. conditions. This implies the series in (2.2) left side can be derived term by term

$$-\sum_{n=1}^{\infty} u_n \phi_n''(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \tag{2.4}$$

Using that

$$\phi_n''(x) = -\lambda_n \phi_n(x). \qquad (3.1)$$

in (2.4) leads to

$$-\sum_{n=1}^{\infty} u_n(-\lambda_n \phi_n(x)) = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

or

$$\sum_{n=1}^{\infty} u_n \lambda_n \phi_n = \sum_{n=1}^{\infty} f_n \phi_n \qquad (3.2)$$

Using orthogonality,

$$\sum_{n=1}^{\infty} u_n \lambda_n \int_0^L \phi_n(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} f_n \int \phi_n(x)\phi_m(x)\,dx$$

$$\Rightarrow \quad u_n \lambda_n \int_0^L \overset{\neq 0}{\phi_n^2(x)}\,dx = f_n \int_0^L \overset{\neq 0}{\phi_n^2(x)}\,dx$$

$$\Rightarrow \quad \boxed{u_n = \frac{f_n}{\lambda_n} = \frac{f_n}{\frac{(n\pi)^2}{L^2}} = \frac{L^2 f_n}{(n\pi)^2}} \qquad (3.3)$$

Therefore,

$$\boxed{u(x) = \sum_{n=1}^{\infty} \frac{L^2 f_n}{(n\pi)^2} \sin \frac{n\pi}{L} x.} \qquad (3.4)$$

Now,
$$f_n \equiv \frac{2}{L}\int_0^L f(x_0)\sin\frac{n\pi}{L}x_0\,dx_0 = 1\Big/\underbrace{\int_0^L \sin^2\!\left(\frac{n\pi}{L}x\right)dx}_{=\,L/2}$$

$\therefore$
$$u(x) = \frac{2}{L}\int_0^L f(x_0)\left(\sum_{n=1}^{\infty}\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x_0\right)\sin\left(\frac{n\pi}{L}x\right)\right)dx_0$$

or
$$u(x) = \int_0^L f(x_0)\underbrace{\left(\frac{2}{L}\sum_{n=1}^{\infty}\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x_0\right)\right)}_{\equiv\; G(x,x_0).}dx_0$$

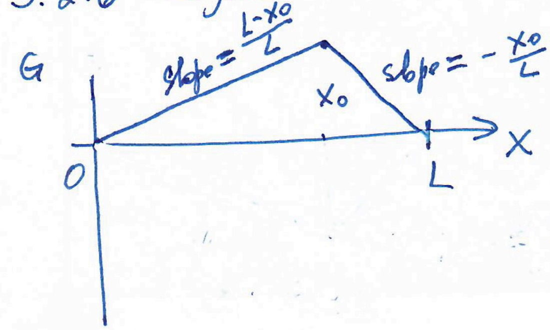
or
$$u(x) = \int_0^L G(x,x_0)\,f(x_0)\,dx_0 \rightarrow \text{(obtained using Green's formula)}$$

where
$$G(x,x_0) = 2L\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x_0\right)$$

Compare with
$$G(x,x_0) = \begin{cases} \dfrac{x_0(L-x)}{L}, & x_0 < x \\[2ex] \dfrac{x(L-x_0)}{L}, & x_0 > x. \end{cases}$$

Problem 5.2.6 Logan's book

Graphic:



The Method of eigenfunction expansion for Green's functions.

For this part review Sturm-Liouville Theorem first.

Consider the Sturm Liouville problem:

$$\begin{cases} Au \equiv -(pu')' + qu = f(x), & a<x<b. \quad (17.1) \\ B_1 u(a) \equiv \alpha_1 u(a) + \alpha_2 u'(a) = 0, & \\ B_2 u(b) \equiv \beta_1 u(b) + \beta_2 u'(b) = 0 & \end{cases}$$

$$\begin{cases} \text{Dirichlet}: & \alpha_2 = 0, \beta_2 = 0 \\ \text{Neumann}: & \alpha_1 = 0, \beta_1 = 0. \\ \text{Robin}: & \alpha_1, \alpha_2, \beta_1, \beta_2 \neq 0 \end{cases}$$

The related eigenvalue problem is given by

$$\begin{cases} A\phi = \lambda\phi, & a<x<b, \quad (17.2) \\ B_1\phi(a) = 0, & B_2\phi(b) = 0 \quad (17.3) \end{cases}$$

We know from our previous discussions that (17.2), (17.3) has infinitely many eigenvalues

$\lambda_n \longrightarrow$ and corresponding eigenfns. $\phi_n(x)$. $n = 1, 2, \ldots$

The set $\{\phi_n\}_{n=1}^{\infty}$ form a basis in $L^2[a,b]$.

and we can assume that the solution of (17.1).

Can be written as $\quad u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$

c_n's are unknowns.

$f(x)$ can also be written as

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \, , \quad \text{where}$$

$$f_n = \int_a^b f(\xi)\, \phi_n(\xi)\, d\xi$$

This is assuming $\{\phi_n\}$ is an <u>orthonormal</u> set.

Substitution into (17.1).

$$Au = f.$$

$$A\left(\sum_1^{\infty} c_n \phi_n(x)\right) = \sum_1^{\infty} f_n \phi_n(x)$$

$$\Rightarrow \quad \sum_1^{\infty} c_n A(\phi_n(x)) = \sum_1^{\infty} f_n \phi_n$$

$$\Rightarrow \quad \sum_1^{\infty} c_n \lambda_n \phi_n = \sum_1^{\infty} f_n \phi_n$$

$$\Rightarrow \quad \boxed{c_n \lambda_n = f_n} \qquad (18.1)$$

Assuming all $\lambda_n \neq 0$ $\Rightarrow$ The original problem has a unique solution.

$$\boxed{c_n = \frac{f_n}{\lambda_n}} \Rightarrow u(x) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n} \phi_n(x)$$

Replacing, f_n by its corresponding integral expression

$$u(x) = \sum_{1}^{\infty} \frac{1}{\lambda_n} \left(\int_a^b f(\xi)\, \phi_n(\xi)\, d\xi \right) \phi_n(x)$$

interchanging Summation and integration.

$$u(x) = \int_a^b \underbrace{\left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\, \phi_n(\xi)}{\lambda_n} \right)} f(\xi)\, d\xi$$

$$\Rightarrow \quad u(x) = \int_a^b g(x,\xi)\, f(\xi)\, d\xi$$

Where

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\, \phi_n(\xi)}{\lambda_n} .$$

Green's function in terms of the eigenvalues and eigenfunctions.

Rmk: 1) Green's function does not exist if for some n_0 $\lambda_{n_0} = 0$. It means if the related eigenvalue problem has $\lambda = 0$ as an eigenvalue.

2) Use this part Combined with the previous lecture to Solve problem 2.6. Logan.